Asymptotic Liapunov Exponents Spectrum For An Extended Chaotic Coupled Map Lattice

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Abstract

The scaling hypothesis for the coupled chaotic map lattices (CML) is formulated. Scaling properties of the CML in the regime of *extensive* chaos observed numerically before is justified analytically. The asymptotic Liapunov exponents spectrum for coupled piece-wise linear chaotic maps is found.

1 Introduction

1.1 The Liapunov exponents spectrum

Partial differential equations describing continuous models and real systems can be discretized into a system of coupled maps called as coupled map lattices (CML). In the last fifteen years, a close attention has also been drawn to these models in virtue of studies of the generic properties of spatiotemporal chaos, [1]-[2]. Nevertheless, by this time, just a few rigorous analytical results on the chaotic dynamics observed in CML have been achieved, [3]-[6].

Chaotic behavior in extended systems is typically incoherent, [12]. The problem of stability in lattice dynamical systems has been extensively discussed in [7]. The long-time limit along orbit of the rate of separation of points in the tangent motion is given by the Liapunov exponents spectrum [8],

$$\lambda_k = \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} \log \left|(DF^t)\right|_k , \tag{1}$$

where (DF^t) is the Jacobian matrix evaluated on the orbit F^t. The notation $|\dots|_k$ denotes the k-th eigenvalue of the matrix (DF), ordered in such a way that $|(DF)|_{k-1} \geq |(DF)|_k \geq |(DF)|_{k+1}$. The spectrum (1) characterizes the stability of motion in the phase space. If all exponents λ_k are negative the trajectory has an attractive fixed point. If the maximal value of the spectrum $\lambda_0 = 0$, the attractor is a limit cycle, whereas positive exponents correspond to diverging orbits.

A simple example of CML is given by the 1-dimensional chain

$$F(c_x^t) \equiv c_x^{t+1} = (1-2\varepsilon) f(c_x^t) + \varepsilon \left[f(c_{x-1}^t) + f(c_{x+1}^t) \right], \quad x \in \mathbb{Z}, \quad t \in \mathbb{Z}_+, \tag{2}$$

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of loosely coupled ($\varepsilon < 1/4$) piecewise linear maps

$$f(c_x^{t+1}) = rc_x^t \mod 1, \tag{3}$$

which is known to induce the chaotic behavior as the value of the control parameter becomes $r \geq 1$. Space-time chaos properties in chains of weakly interacting hyperbolic maps has been considered in [9].

The Liapunov exponent of the solitary uncoupled map (3) is known exactly as a function of the control parameter r, $\lambda_0 = \log r$. Furthermore, since the correspondent Jacobian in (2) is a $N \times N$ tridiagonal matrix, its eigenvalues can be computed explicitly for any finite number of sites N provided the boundary conditions (c_{-1} and c_{N+1}) are known, [10]. For example, chosen the zero valued boundary conditions, $c_{-1} = c_{N+1} = 0$, one obtains the following Liapunov exponents spectrum,

$$\lambda_k = \lambda_0 + \log\left(1 - 2\varepsilon + 2\varepsilon\cos\frac{\pi k}{N+1}\right), \quad k = 1 \ldots N. \tag{4}$$

For any finite N, the spectrum (4) is bounded from above by the maximal value coinciding with the Liapunov exponent of uncoupled maps λ_0. Either of the spectral values $\lambda_k < \lambda_0$ depend both on the boundary conditions chosen and the actual value of the coupling strength parameter ε. In the extended limit, $N \to \infty$, the boundary conditions can be assumed as an asymptote $\lim_{N\to\infty} c_{\pm N} = 0$, and the spectrum (4) turns to a band. In actual numerical simulations of the extended limit, the boundary conditions are usually taken as periodic.

More realistic models for partial differential equations refer to more complicated coupled maps lattices for which the Liapunov exponents spectra have not been found. Therefore, one is interested in methods which can yield at least qualitative information on them.

The Liapunov exponents spectra of CML in the extended limit, $N \to \infty$, is of special interest, [12]. Intuitively, if the system is large enough, subsystems that are sufficiently separated in space become decorrelated and so contribute to the fractal dimension in proportion to their number. This phenomenon usually called as an *extensive* chaos, [11], is numerically observed for lattice sizes larger than some threshold value depending on the model, [6].

For large lattices, the number of positive Liapunov exponents $\lambda_k > 0$ scales like the lattice size

$$N_{\lambda_k>0} \sim N/\mathcal{U}, \tag{5}$$

where N is the volume of a large albeit finite piece $\mathbb{L}$ of the lattice, and $\mathcal{U}$ is said to be the volume of a typical statistically independent block. In accordance with numerical simulations data, $\mathcal{U}$ depends substantially on the particular model while the relation (5) is believed to hold generally. The result (5) has been observed numerically for some simple coupled map lattices (see [13] and references therein for a review). Nevertheless, up to our knowledge, the scaling (5) has not been justified analytically, in particular, the meaning of the statistically independent block $\mathcal{U}$ is still vague.

Another fascinating phenomenon observed in the chaotic coupled map lattices [14]-[12] is a non-trivial collective behavior ("hidden coherence", [15]), where there is apparently a unique nice invariant measure, [26], but nevertheless non-uniqueness of nice measures on the space-time configurations, because a two-cycle or more complicated dynamic behavior occurs in time. The time periodic behavior of measures has been discussed in [16] ("asymptotic periodicity").

The chaotic dynamics in CML lacks of mathematically rigorous results. Nevertheless, some "physical" approaches borrowed from statistical physics and quantum mechanics can be applied to the theory of chaotic spatially extended dynamical systems. An essential problem of the "physical" point of view is that one implies the nice invariant measure (or nice measures if the system is subjected to a phase transition) is defined. In general, coupled maps are not the case except the coupling parameter ε is not vanishingly small, [5].

Our example of coupled maps chain (2) has an invariant measure defined as a Gibbs state even if ε is not small, [17]. The non-trivial collective behavior ("hidden coherence") is not actually captured by the present paper. At the same time, while computing the generalized entropy K_q and the partition function $\mathbf{Z}$, we shall perform the summations $\sum_{\{\xi\}}$ over "all configurations" $\{\xi\}$. The summation technique that we use relies upon the symmetry properties of the coupled chaotic maps chain and is known as a *collective coordinates* method. Proceeding with computations, we replace $\sum_{\{\xi\}}$ with an integration over an infinite set of continuous real valued positive variables. This stroke also implies that there is a number of nice measures on the space-time configurations.

1.2 The scaling hypothesis for coupled map lattices

The *scaling* hypothesis in thermodynamics has been an important milestone in formulation of the modern phase transitions theory. It is based on two fundamental principles of non-equilibrium thermodynamics:

i) while interesting in critical phenomena, one can substitute a true micro-model for a *fluctuation* one;

ii) in the regime of strongly developed fluctuations, the initial values are of no consequences for the system behavior.

For the Ising model, the scaling hypothesis put forward by Kadanoff [18] leans upon the "blocks" procedure, in which the interactions of spins have been replaced for the interactions of "spin blocks". It consists of two propositions:

j) the Hamiltonian of "spin blocks" interaction has just the same form as the Hamiltonian of spin interaction with parameters changed;

jj) the dependence of parameters on the block size L has a scaling form as $L \to \infty$.

For the uncoupled maps having a "short range time memory", the scaling hypothesis has been propounded first in [19], [20] and then used in [21].

We will follow a track by considering fluctuating effective quantities analogous to the Liapunov exponents in finite time segments, $n \gg 1$. The "effective Liapunov exponent" λ^n define

the rate of separation of points in tangent motion in finite time. They fluctuates around the true Liapunov exponent λ having the fluctuation range Λ, [21]:

$$\lambda = \lim_{n\to\infty} \int d\Lambda\ \rho_n(\Lambda)\lambda^n, \tag{6}$$

in which $\rho_n(\Lambda)$ is the probability distribution to observe fluctuation Λ in n time steps. In the limit $n \to \infty$, λ^n form a micro-canonical ensemble $\lim_{n\to\infty} \rho_n(\Lambda) = \delta(\Lambda)$.

An example of $\rho_n(\Lambda)$, one can consider the Fig.1 where we have plotted the data of 5000 numerical simulations for the uncoupled solitary map (3) at $r = 1.6$, for the effective Liapunov exponent λ^n, $n = 700$. The graph shows the probability distribution to observe deviations from the true Liapunov exponent $\lambda_0 \approx 0.47\ldots$. When n growths up, the bell-shaped profile becomes sharper.

Being a product of n Jacobi matrices, the effective rates λ^n, for the maps with a short-range time memory, behave like averages of n random variables correlated only over short time. Therefore, their values, for n large, do not depend on the particular choice of initial conditions.

The scaling hypothesis formulated in [19] for such maps claims on the *self-similarity* property for the microscopical entropy, $S = \log \rho(\Lambda)$ in a long however finite time string, $n >> 1$, in the chaotic regime:

$$\rho_n(\Lambda) \propto \exp[-S], \quad S = ng(\Lambda) \tag{7}$$

where $g(\Lambda)$ is a "scaling function" (a "free energy") pertinent to the map, [19].

In the Fig.2, we have pictured out the ratio of scaling functions g_{n_1}/g_{n_2} via Λ computed numerically for the map (3) in the sequences of $n_1 = 600$ and $n_2 = 1200$ time steps. The picture data show that there is an interval of deviations ($\simeq [-0.1, 0.1]$) within which the scaling function $g(\Lambda)$ is almost *universal* at least in the first 1200 time steps.

We note that in accordance with (7), given a smooth function $F(\Lambda)$, the relevant observable is $\langle F\rangle = \int d\Lambda\ \rho(\Lambda)F(\Lambda)$. Since n is a large number, one infers that the main contribution to this integral comes from the saddle-point configuration Λ_* such that $g(\Lambda_*)$ is the maximal g.

Indeed, this hypothesis breaks down for maps with strong long-time correlations, when a minor change of initial conditions can stir the behavior even in a very long time segments.

In the present paper, we extend the scaling hypothesis to the case of coupled maps lattices. We search for the statistics of fluctuating quantities λ_k^n analogous to the true Liapunov exponents λ_k in a long albeit finite time. We consider the extended lattices ($N \to \infty$) of coupled maps with a short range time memory. By the way, the boundary conditions are of no consequences for the system behavior in "thermodynamic limit".

Therefore, following [18] and resting on the numerical simulations data cited in [13], we propose that in the chaotic regime the "free energy" function $g_{\mathcal{U}}(\Lambda)$ pertinent to the CML is universal up to the statistically independent block $\mathcal{U}$, so that the probability to observe a deviation of range $\Lambda = |\lambda_k^n - \lambda_k|$ in the thermodynamic limit equals to

$$\rho(\Lambda) \propto \exp[-S], \quad S = nN_{\mathcal{U}}g_{\mathcal{U}}(\Lambda). \tag{8}$$

Furthermore, we propose that in the chaotic regime the scaling function pertinent to the coupled maps lattice, $g_{\mathcal{U}}$, up to a change of parameters equals to that one for the solitary uncoupled map, g.

In thermodynamic limit, according to (8), the most important contribution to observables comes from saddle-points configurations Λ_* (provided they exist) maximizing the scaling function $g_{\mathcal{U}}$.

1.3 Connections of a loosely coupled maps chain to a quantum mechanical system with a double-well potential

Let us remark on a semblance between the statistics of effective Liapunov exponents of the simplest loosely coupled ($\varepsilon \ll 1$) maps chain (2) in thermodynamic limit and of the quantum mechanical system with a double-well potential in the infinite β limit (i.e., in small temperature limit, $T \to 0$) [22].

For $\beta \gg 1$, the main contribution to the partition function of the quantum mechanical problem comes from the *instanton* (or saddle-points) configurations which renders the action stationary. In the limit $\beta \to \infty$, a linear combination of instanton solutions plays a role. The relevant contributions do not correspond to solutions of the classical equations of motion since these equations are non-linear. However, in the limit when all instantons are largely separated, such a combination also renders the action almost stationary since the relevant corrections coming from each instanton are exponentially small in the separation.

To properly study the energy spectrum of the quantum mechanical problem at $T \to 0$, first, one has to introduce a set of collective variables (or coordinates) which describe fluctuations in distances between the instantons, and, second, to fix out their values in such a way that minimizes the variation of the action.

In the present paper, we are going to apply this method to compute the asymptotic of a partition function $\mathbf{Z}$ over the space of possible configurations $\{c_x^t\}$ generated by the chain of coupled maps (2).

In Fig. 3.1, we have plotted the results of 10000 numerical simulations for the chain of $N = 8$ coupled maps (the value of coupling strength parameter has been taken as $\varepsilon = 0.02$) supplied by the zero valued boundary conditions. This graph shows the probability distribution profile to observe the effective Liapunov exponents λ^n in the interval $[0.44 \ldots 0.48]$ in $n = 1000$ iterations. One can see that almost all values λ^n drop around the true Liapunov exponents laying within this interval, $\lambda_0 \simeq 0.4700\ldots$, $\lambda_1 \simeq 0.4660\ldots$, and $\lambda_2 \simeq 0.4548\ldots$. The quota of λ^n arising apart from the close vicinities of Liapunov exponents λ_i decreases very fast as n grows up.

In Fig. 3.2, we have drawn the distribution profile at $n = 2500$ time steps. For $n \gg 1$, the dispersion δ becomes much smaller than the distances $\Delta\lambda_i$ between peaks, $\delta \ll \Delta\lambda_i$. This means that fluctuations which tend to change distances between Liapunov exponents induce just a vanishingly small variation of g at long time segments.

We note that the thermodynamic limit when fluctuations of λ_k^n are faded out, plays the role of the infinite separation limit for instantons in the quantum well problem mentioned above.

The plan of the paper follows. In the Sec. 2, leaning on the thermodynamic formalism as applied to CML, we explain what kind of information on the Liapunov exponents spectrum one could obtain from the relevant partition function $\mathbf{Z}$. Then, in Sec. 3 and 4, we compute $\mathbf{Z}$ using the scaling hypothesis formulated in the previous subsection (1.2) and following methods developed in quantum mechanics with respect to the double-well potential problem [22]. Namely, in the Sec. 3, we compute the scaling function $g(\Lambda)$ (the free energy of the microscopical interaction) for the map (3) coupled merely to the nearest neighbors. In the Sec. 4, we introduce the block scaling function $g_{\mathcal{U}}(\Lambda)$ (the block interaction) and then compute the asymptotic of $\mathbf{Z}$ in thermodynamic limit utilizing the collective variables method. We conclude in the Sec. 5.

2 Thermodynamic formalism and the Liapunov exponents

Thermodynamic formalism relies on the existence of a symbolic representation for the dynamics. The idea to study this representation via Gibbs states for the $d+1$-dimensional system goes back to [23] and [24].

Let us suppose that one can split the phase space of CML $L \subset \mathbb{Z}^d$ into disjoint boxes b_i of size a. We shall call as an *orbit* generated by the coupled map in site x_i the sequence of boxes, $\{b_1, \dots b_j, \dots b_n\}$, in which the coupled map takes values at consequent time steps $j = 1 \dots n$. We shall call as a *configuration* ξ_L^n the set of orbits,

$$\xi_L^n = \{\{b_1, \dots b_n\}_{x_i}, x_i \in L\},$$

in the finite piece $L \subset \mathbb{Z}^d$. The thermodynamic formalism comes about by asking for a conditional probability $P\left(\xi_L^n | \, \xi_{\mathbb{Z}^d \setminus L}^n\right)$ to observe the particular configuration ξ_L^n given the configuration in the complement $\xi_{\mathbb{Z}^d \setminus L}^n$. This probability looks like an expectation in the canonical ensemble,

$$P\left(\xi_L^n | \, \xi_{\mathbb{Z}^d \setminus L}^n\right) = \mathbf{Z}^{-1} \exp\left[-\beta \Phi\left(\xi_L^n | \, \xi_{\mathbb{Z}^d \setminus L}^n\right)\right], \tag{9}$$

in which the normalization factor (the partition function, in terms borrowed from statistical physics) is

$$\mathbf{Z}(L; \xi_{\mathbb{Z}^d \setminus L}^n) = \sum_{\{\xi_L^n\}} \exp\left[-\beta \Phi\left(\xi_L^n | \, \xi_{\mathbb{Z}^d \setminus L}^n\right)\right] \tag{10}$$

(the sum is taken over all configurations ξ_L^n depending upon possible configurations $\xi_{\mathbb{Z}^d \setminus L}^n$ in the complement). Here, $\beta > 0$ is an inverse temperature parameter representing a "heat bath" of coupling maps.

The interaction potential Φ plays the role of a Hamiltonian defined for the finite sublattice L by

$$\Phi\left(\xi_L^n | \, \xi_{\mathbb{Z}^d \setminus L}^n\right) = \sum_{n \in \mathbb{Z}_+} \log \left| \det(DF)\left(\xi_L^n | \, \xi_{\mathbb{Z}^d \setminus L}^n\right) \right|. \tag{11}$$

For infinitely long time strings, it depends upon the configuration $\xi_{\mathbb{Z}^d}$ in the whole lattice. The Jacobi matrix (DF) can be found in the form $(DF) = B(I-W)$ where B is the diagonal part of the matrix which provides a point-wise contribution from maps with no coupling, and W comes from coupling. The coupling W possesses the translation invariance property. Therefore, the potential (11) can be extended naturally to all subsets of the lattice $\mathbb{Z}^d$ by translation invariance.

For future convenience, we use the identity $\det \equiv \exp \mathrm{Tr} \log$ to rewrite the potential (11) in the following form

$$\Phi = \sum_{n\in\mathbb{Z}_+} Tr \ \left[\log(DF)\left(\xi^n_L|\, \xi^n_{\mathbb{Z}^d\setminus L}\right)\right], \tag{12}$$

which has been discussed extensively in [3]-[25], and [26]. Due to expansion $\log(DF) = \log B + \log(I-W)$, the Hamiltonian (12) can be divided in two folds: $\Phi_L = \Phi_0 + U$, where Φ_0 is a "free" Hamiltonian relevant to the uncoupled maps and the coupling "interaction" U.

Using the definition of effective Liapunov exponents, λ^n_k, one can express the partition function (10) as follows,

$$\mathbf{Z} = \lim_{n\to\infty} \sum_{\{\lambda^n_k\}} \exp\left(n\beta \mathrm{Tr}[\lambda^n_k]\right) = \exp\left(\beta \mathrm{Tr}[\lambda_k]\right) \tag{13}$$

where the sum $\sum_{\{\lambda^n_k\}}$ is taken over the ensemble of fluctuating quantities λ^n_k. For the uncoupled maps, the Eq. (13) yields $\lambda = \lim_{n\to\infty} \log \mathbf{Z}/n\beta$. However, for coupled map lattices, one can just obtain merely the sum of positive Liapunov exponents $\mathrm{Tr}[\lambda_k]$.

It is essential to make use of the scaling hypothesis (Sec. 1.2) while searching for the Liapunov exponents. In particular, it implies (see Sec. 4) that

$$\mathrm{Tr}[\lambda_k] \sim_{N\gg 1} N. \tag{14}$$

Therefore, at least for large N, the values of the Liapunov exponents λ_N can be found by the simple subtraction, $\lambda_N = \mathrm{Tr}[\lambda_{N-1}] - \mathrm{Tr}[\lambda_N]$. Furthermore, the maximal value in the spectrum, λ_0 equals to the Liapunov exponent of the uncoupled map and, at least for the chain (2), is also known.

However, since the relation (14) is formally valid only for $N \gg 1$, the spectral values λ_k, $k \sim O(1)$, are still uncertain. The numerical simulations of systems demonstrating the extensive chaotic behavior (i.e., scaling) give an evidence (see for example, [6]) in favour of that the threshold lattice size is typically $N \simeq 8$.

3 Scaling function $g(\Lambda)$ for the map coupled merely to the nearest neighbors

The scaling function $g(\Lambda)$ plays a role of free energy in the ensemble of fluctuating effective Liapunov exponents. In the limit $n \to \infty$, it can be related to the generalized entropies [19] K_q defined by

$$K_q(\Lambda) = \frac{1}{q-1} \lim_{n\to\infty} \frac{1}{n} \log \sum_{\{\lambda^n\}} \rho_n(\Lambda)^q, \tag{15}$$

via Legendre transforms, [19],

$$K_q(\Lambda) = \frac{1}{q-1}\left[\Lambda q - g(\Lambda)\right], \quad g(\Lambda) = \chi(q) - q\frac{\partial}{\partial q}\chi(q), \quad \chi(q) \equiv (q-1)K_q. \tag{16}$$

It has been extensively discussed in a literature that $\lim_{q\to 0} K_q$ and $\lim_{q\to 1} K_q$ are the topological and metric entropies, respectively. An example of computation of $g(\Lambda)$ has been given in [19] for the simple uncoupled two-dimensional "baker's" map.

Let us choose the site $\mathbf{0}$ as an origin and introduce a point-wise Hamiltonian

$$\Phi_n^\varepsilon(\mathbf{0}) = \lambda_{\varepsilon 0} + U_n^\varepsilon(\mathbf{0}, V_\mathbf{0}), \tag{17}$$

in which the interaction potential $U_n^\varepsilon(\mathbf{0}, V_\mathbf{0})$, and $V_\mathbf{0}$ is a nearest neighborhood of $\mathbf{0}$. In the present section, we suppose that there is no interactions between $V_\mathbf{0}$ and remainder of the lattice.

The free point-wise Hamiltonian equals to the logarithm of diagonal elements of the Jacobi matrix, $\lambda_{\varepsilon 0} = \log\left(r(1-2\varepsilon)\right)$. The potential $U_n^\varepsilon(\mathbf{0}, V_\mathbf{0})$ describes the instantaneous fluctuations of the effective Liapunov exponent. It comprises of a sum of contributions coming from the neighboring sites $\mathbf{y} \in V_\mathbf{0}$,

$$U_n^\varepsilon(\mathbf{0}, V_\mathbf{0}) = \sum_{\mathbf{y}\in V_\mathbf{0}} u_{n\mathbf{0}}^\varepsilon(\mathbf{y}).$$

The relevant point-wise generalized entropy takes the form:

$$K_q(\varpi_\varepsilon) = \frac{1}{q-1}\lim_{n\to\infty}\frac{1}{n}\log\sum_{\{\lambda^n\}}\exp\left[\beta q\varpi_\varepsilon - \beta q\sum_{\mathbf{y}\in V_\mathbf{0}} u_{n\mathbf{0}}^\varepsilon(\mathbf{y})\right], \tag{18}$$

in which $\varpi_\varepsilon \equiv \lambda_0 - \lambda_{\epsilon 0}$, and $\sum_{\{\lambda^n\}}$ denotes the sum over all possible orbits $\{\lambda^n\}$. Note, that the first term in the exponent, $\mu \equiv \exp\left[\beta q\varpi_\varepsilon\right]$ persists for any orbit. It plays a role of the "fugacity" of the fluctuation "gas".

To perform the summation in (18) properly, we note that the site $\mathbf{0}$ and its nearest neighbors $V_\mathbf{0}$ form an isolated system. Therefore, any fluctuation risen in $\mathbf{0}$ can be represented as a vector in the V-dimensional space of particular contributions from the neighboring sites, $\mathbb{U} \equiv \{u_{n\mathbf{0}}^\varepsilon(\mathbf{y})\}$. Since the entropy of closed system does not change in time, the sum $\sum_{\{\lambda^n\}}$ can be interpreted as an integral over the set of linear transformations $\mathbb{U} \to \mathbb{U}$ which render the entropy (18) unchanged.

The transformations we are interested in is apparently defined by the set of positive parameters $\varphi_\mathbf{y}^n > 0$, such that

$$\beta = \sum_{\mathbf{y}\in\mathcal{V}_0}\varphi_y^n, \ \forall n. \tag{19}$$

Choosing at each time step $V-1$ values $\{\varphi_y^n\}$ at random, one generates a particular orbit $\{\lambda^n\}$. Therefore, one can replace the sum over possible orbits in (18) with integrations over the real positive variables φ_y^n,

$$K_q(\varpi_\varepsilon) = \frac{1}{q-1}\lim_{n\to\infty}\frac{1}{n}\log\sum_{n\in\mathbb{Z}_+}\left[\frac{1}{n}\int_{\varphi_y^n>0}\prod_{n\in\mathbb{Z}_+}\prod_{\mathbf{y}\in V_\mathbf{0}} d\varphi_y^n \times\right. \tag{20}$$

$$\times\, e^{nq\left[\beta\varpi_\varepsilon - 1/n\sum_{\mathbf{y}\in V_{\mathbf{0}}}\varphi_y^n u^\varepsilon_{n\mathbf{0}}(\mathbf{y})\right]}\cdot\delta\left(\beta-\sum_{\mathbf{y}\in V_{\mathbf{0}}}\varphi_y^n\right)\Bigg].$$

The factor $1/n$ before the integral over φ_y^n in (20) arises because the result of integration has to be invariant under a cyclic permutation of φ_y^n-variables in time.

After the integration over φ_y^n variables [27], in (20), one obtains

$$K_q(\varpi_\varepsilon)=\frac{1}{q-1}\lim_{n\to\infty}\frac{1}{n}\log\sum_{n\in\mathbb{Z}_+}\frac{1}{2\pi ni}\int_{-\eta-i\infty}^{-\eta+i\infty}\frac{e^{\beta(s+n\varpi_\varepsilon)}}{\mathrm{P}_V^{(n)}(s,q)}ds, \tag{21}$$

in which $\mathrm{P}_V^{(n)}(s,q)=\prod_{n\in\mathbf{Z}_+}\prod_{\mathbf{y}\in V_{\mathbf{0}}}\left(s-qu^\varepsilon_{n\mathbf{0}}(\mathbf{y})\right)$ is a the polynomial of the order V. For piecewise linear maps, one can proceed further since the slope of map is a piecewise constant, and the values $u_{n\mathbf{0}}(\mathbf{y})$ in the polynomial $\mathrm{P}_V^{(n)}(s,q)$ does not depend on n.

The sum $\sum_{n\in\mathbb{Z}_+}$ therefore can be easily performed,

$$K_q(\varpi_\varepsilon)=\frac{1}{q-1}\log\left[\frac{1}{2\pi i\beta}\int_{-\eta-i\infty}^{-\eta+i\infty}ds\beta e^{-\beta s}\log w(s,\varpi_\varepsilon)\right] \tag{22}$$

where we have defined

$$w(s,\varpi_\varepsilon)=1-\frac{e^{-\beta\varpi_\varepsilon}}{\prod_{\mathbf{y}\in V_{\mathbf{0}}}\left(s-qu^\varepsilon_{n\mathbf{0}}(\mathbf{y})\right)}.$$

The remaining integration in (22) is performed by parts,

$$K_q(\varpi_\varepsilon)=\frac{1}{q-1}\log\left[\frac{1}{2\pi i\beta}\int_{-\eta-i\infty}^{-\eta+i\infty}ds\ \beta e^{-\beta s}\frac{w'(s,\varpi_\varepsilon)}{w(s,\varpi_\varepsilon)}\right].$$

The contour can be deformed in the half-plane $\Re(s)\le 0$ to enclose the poles of the integrand given by the solutions of the equation

$$\prod_{\mathbf{y}\in V_{\mathbf{0}}}\left(s-qu^\varepsilon_{n\mathbf{0}}(\mathbf{y})\right)=e^{-\beta\varpi_\varepsilon}. \tag{23}$$

The solutions of the equation (23) can be easily found,

$$s=qu^\varepsilon_{n\mathbf{0}}+\exp[-\frac{\beta}{V}\varpi_\varepsilon]. \tag{24}$$

Then, the residue theorem yields:

$$K_q(\varpi_\varepsilon)=\frac{\beta}{q-1}\left[q\frac{\varpi_\varepsilon}{V}+qu^\varepsilon_{n\mathbf{0}}-e^{-\frac{\beta}{V}\varpi_\varepsilon}\right]. \tag{25}$$

Using the Legendre transforms (16), one obtains the relevant scaling function,

$$g(\varpi_\varepsilon)=\beta e^{-\frac{\beta}{V}\varpi_\varepsilon}. \tag{26}$$

The scaling function $g(\Lambda)$ (26) has a typical form pertinent to one-instanton contributions as they appear in different problems in quantum mechanics. We also note the relation between the fugacity of fluctuation gas μ and $g(\Lambda)$, $\mu(\Lambda)=g^V(\Lambda)$.

4 The partition function in the thermodynamic limit

In accordance to the scaling hypothesis, while proceeding to the interactions between blocks, the Hamiltonian of the interaction has to be the same as the microscopic one, except the values of parameters. The general scaling in the system (i.e., the fact that the dependence of parameters on the block size L has a scaling form as $L \to \infty$) then follows as a natural consequence from the latter preposition [28].

Proceeding from (26), for the statistically independent block $\mathcal{U}$ (which constructive definition we give below), one obtains the relevant interaction,

$$g_{\mathcal{U}}(\Lambda) = \exp\left[-\frac{\beta}{\mathcal{U}}\Lambda\right]. \tag{27}$$

It differs from (26) just on the volume $\mathcal{U}$.

Let us consider a finite piece $\mathbb{L}$ of the lattice $\mathbb{Z}^d$. We assume that $\mathbb{L}$ is large enough to comprise of many statistically independent blocks $\mathcal{U}$. For each site $x \in \mathbb{L}$, we assign an inverse temperature parameter $\theta_x > 0$. These parameters is to be subjected to two obvious constrains expressing the idea of statistical independency of the block from the rest of lattice:

1. The statistically independent block has to be in thermodynamical equilibrium with the remainder of lattice. Therefore,
$$\beta = \sum_{s=1}^{N_{\mathcal{U}}} \theta_{x_s} \tag{28}$$
where the summation is going over the sites belonging to the same block $\mathcal{U}$.

2. All parts of the sublattice $\mathbb{L}$ should be in thermodynamic equilibrium with respect to each other,
$$\frac{\beta}{\mathcal{U}} = \frac{1}{N_{\mathcal{U}}} \sum_{s=1}^{N} \theta_{x_s}, \tag{29}$$
in which $\mathcal{U}$ denotes the number of lattice sites belonging to one statistically independent block, N is a total number of sites in $\mathbb{L}$, and $N_{\mathcal{U}}$ is a total number of statistically independent blocks in $\mathbb{L}$.

Pursuant to (8), the block-wise partition function has a scaling form,

$$\mathbf{Z} = \sum_{\{\mathcal{U}\}} \exp\left[-\beta N_{\mathcal{U}} g_{\mathcal{U}}(\Lambda)\right] \tag{30}$$

where the summation is performed over all statistically independent (in the sense of constrains 1.) and 2.)) collections of sites $\{\mathcal{U}\}$. To compute this sum in the thermodynamic limit ($N_{\mathcal{U}} \to \infty$), we employ the collective variables method again.

In terms of the θ_x-variables, the block-wise scaling function reads as

$$g_{\mathcal{U}}(\Lambda) = \exp\left[-\frac{\Lambda}{N_{\mathcal{U}}} \sum_{s=1}^{N} \theta_{x_s}\right],$$

and therefore,

$$\mathbf{Z} = \sum_{k=1}^{N_{\mathcal{U}}} \left[\frac{1}{k} \int_{\theta_{x_s}>0} \prod_{s=1}^{k} d\theta_{x_s} \ \exp\left[-\beta N_{\mathcal{U}} e^{-\frac{\Lambda}{N_{\mathcal{U}}} \sum_{s=1}^{N} \theta_{x_s}} \right] \delta \left(\beta - \sum_{s=1}^{N_{\mathcal{U}}} \theta_{x_s} \right) \right] \tag{31}$$

where the factor $1/k$ in front of the integral is risen due to cyclic permutation of θ-variables. We use the integral representation for the $\delta-$function ([27]) and introduce the new auxiliary variable α such that

$$\mathbf{Z} = \sum_{k=1}^{N_{\mathcal{U}}} \left[\frac{1}{2\pi i} \int_{-\eta-i\infty}^{-\eta+i\infty} d\alpha \ e^{\beta\alpha} \sum_{k=1}^{N_{\mathcal{U}}} \frac{1}{k} \left[J(\alpha) \right]^k \right], \tag{32}$$

with the definition:

$$J(\alpha) = \int_0^{\infty} \exp\left[\alpha\theta + \beta N_{\mathcal{U}} e^{\theta\Lambda/N_{\mathcal{U}}} \right] \ d\theta.$$

By giving to α a small negative real part, we have insured the convergence of the integral $J(\alpha)$. By means of the change of variables: $\tau \equiv \beta N_{\mathcal{U}} e^{-\theta\varpi_\varepsilon/N_{\mathcal{U}}}$, one obtains,

$$J(\alpha) = \int_0^{\beta N_{\mathcal{U}}} \frac{d\tau}{\tau} \left(\frac{\beta N_{\mathcal{U}}}{\tau} \right)^{\frac{-N_{\mathcal{U}}\alpha}{\Lambda}} e^{-\tau}.$$

In thermodynamic limit $N_{\mathcal{U}} \to \infty$, this integral converges to the $\Gamma-$function,

$$J(\alpha) \sim (\beta N_{\mathcal{U}})^{\frac{N_{\mathcal{U}}\alpha}{\Lambda}} \Gamma\left(\frac{N_{\mathcal{U}}\alpha}{\Lambda} \right).$$

The sum over k in (32) can be performed easily:

$$\mathbf{Z} = \left[\frac{1}{2\pi i} \int_{-\eta-i\infty}^{-\eta+i\infty} d\alpha \ e^{\beta\alpha} \log\left[1 - (\beta N_{\mathcal{U}})^{\frac{N_{\mathcal{U}}\alpha}{\Lambda}} \Gamma\left(\frac{N_{\mathcal{U}}\alpha}{\Lambda} \right) \right] \right]. \tag{33}$$

The integral in (33) can be computed by parts. Due to asymptotic behavior of the $\Gamma-$function, the integral converges, and the contour of integration can be deformed to enclose the poles of the integrand in the half-plane $Re(\alpha) \leq 0$. These poles correspond to the solutions of the equation:

$$(\beta N_{\mathcal{U}})^{\frac{N_{\mathcal{U}}\alpha}{\Lambda}} = \Gamma\left(\frac{N_{\mathcal{U}}\alpha}{\Lambda} \right). \tag{34}$$

In the limit $N_{\mathcal{U}} \to \infty$, solutions of the equation (34) are close to the poles of $\Gamma-$function, i.e. to negative integers.

Since the coupling strength parameter ε is small, one can taylor the $\Gamma-$function in powers of ε to find out the approximate solutions of (34). In the first order in ε, this procedure gives:

$$\frac{\alpha_\ell N_{\mathcal{U}}}{-2\varepsilon} = -\ell, \quad \ell = 1 \ldots N.$$

Finally, one arrives at the spectrum of possible values

$$\alpha_\ell = \frac{2\varepsilon\ell}{N_{\mathcal{U}}}. \tag{35}$$

The asymptotic result for the spectrum of Liapunov exponents follows from (35),

$$\lambda_\ell \simeq_{N\to\infty,\ \varepsilon\to 0} \lambda_0 - 2\varepsilon + \frac{2\varepsilon}{N}\ell + \mathrm{O}(\varepsilon^2), \quad \ell = 1 \ldots N. \tag{36}$$

The spectrum (36) is consistent with the relation (5) observed numerically for wide variety of CML. One can see that for $N \gg 1$ and ε small it resembles closely the spectrum (4).

5 Discussion and Conclusions

In the present paper, we have proposed and analyzed the statistical theory of fluctuations of the effective Liapunov exponents which correspond to the true Liapunov exponents in finite time segments.

We have formulated the scaling hypothesis for the CML and justified the general scaling behavior (5) of the CML in the extensive chaotic regime.

Due to presence of large parameters (n and N), fluctuations of the effective Liapunov exponents can be well described by the saddle-point approximation.

In the framework of thermodynamical formalism in CML, we have computed the scaling function $g(\Lambda)$ for the $1-$dimensional chain of coupled piece-wise linear maps and demonstrated that it has a form typical for the $1-$instanton contribution in the quantum mechanical problem with a double-well potential.

The basic technical point of scaling hypothesis is that the block-wise Hamiltonian appears to have the same form as the micro-interaction. As a direct consequence of this assumption, one arrives at the following scaling properties of the Liapunov exponents spectrum λ_k, for large lattices, $N \gg 1$:

$$N_{\lambda_k>0} \sim N/\mathcal{U}, \quad \text{and} \quad \mathrm{Tr}(\lambda_k) \sim N.$$

They allow to compute the asymptote of the Liapunov exponents spectrum for $N \gg 1$.

We should note that the developed technique can be also applied to the multidimensional coupled piecewise linear maps. However, it probably fails in case of non-linear map lattices. Nevertheless, since most of non-linear maps can be approached by piecewise linear ones, we believe that our technique would also give good approximations to more complicated loosely coupled non-linear map lattices.

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$$\delta\left(\beta - \sum_{y\in\mathcal{V}} \varphi_y^n\right) = \frac{1}{2\pi i}\int_{-\eta-i\infty}^{-\eta+i\infty} ds \exp\left[-s\left(\beta - \sum_{y\in V}\varphi_y^n\right)\right], \quad \eta > 0.$$

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CAPTION FOR FIGURES

Figure 1.

The probability to observe deviations of λ^n, $n = 700$, from the true Liapunov exponent $\lambda_0 \approx 0.47\ldots$ (5000 simulations).

Figure 2.

The ratio of scaling functions g_{n_1}/g_{n_2} via the deviation range Λ computed in the sequences of $n_1 = 600$ and $n_2 = 1200$ time steps.

Figure 3.1

The probability distribution profile to observe the effective Liapunov exponents λ^n in the interval $[0.36\ldots 0.46]$ in $n = 1000$ iterations (10^4 simulations, the chain of $N = 8$ coupled maps, the coupling strength parameter, $\varepsilon = 0.02$).

Figure 3.2

The probability distribution profile to observe the effective Liapunov exponents λ^n in the interval $[0.36\ldots 0.46]$ in $n = 2500$ iterations (10^4 simulations, the chain of $N = 8$ coupled maps, the coupling strength parameter, $\varepsilon = 0.02$).

FIGURE 1

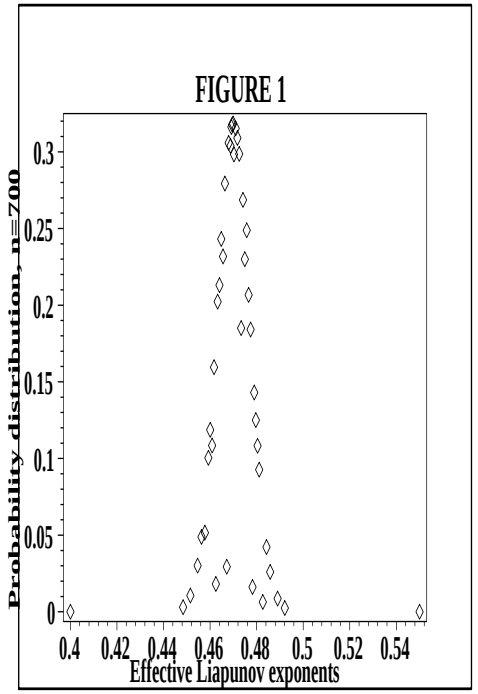

Probability distribution, n=700
0.3
0.25
0.2
0.15
0.1
0.05
0
0.4
0.42
0.44
0.46
0.48
0.5
0.52
0.54
Effective Liapunov exponents

FIGURE 3.1

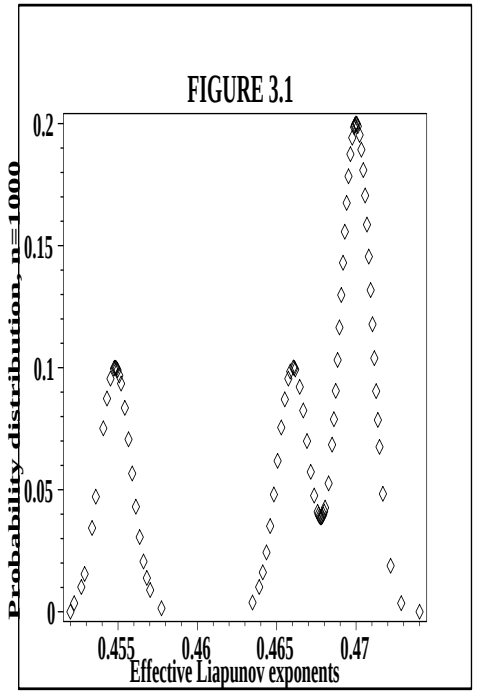

Probability distribution, n=1000
0.2
0.15
0.1
0.05
0
0.455
0.46
0.465
0.47
Effective Liapunov exponents

FIGURE 2

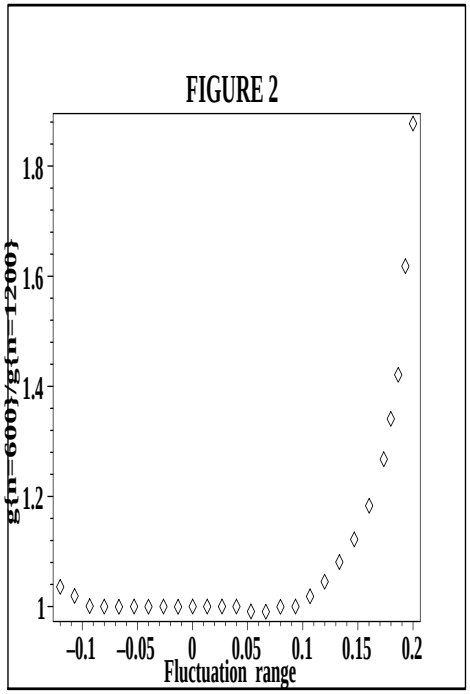
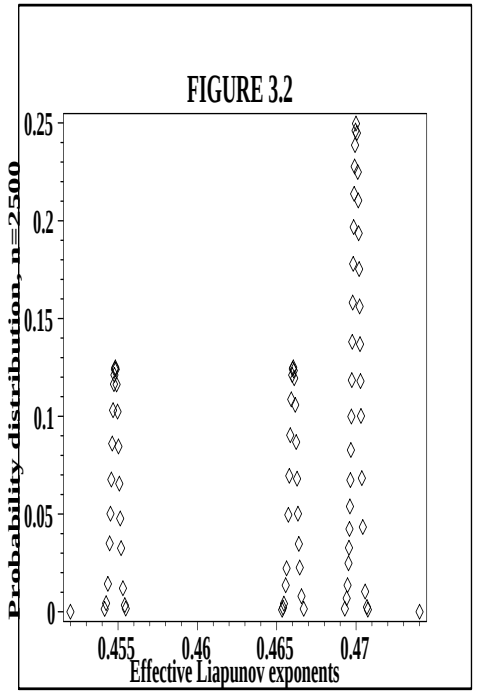

g(n=600)/g(n=1200)
1.8
1.6
1.4
1.2
1
−0.1
−0.05
0
0.05
0.1
0.15
0.2
Fluctuation range

FIGURE 3.2


Probability distribution, n=2500
0.25
0.2
0.15
0.1
0.05
0
0.455
0.46
0.465
0.47
Effective Liapunov exponents